MATH 245 S17, Exam 2 Solutions

1. Carefully define the following terms: free variable, predicate, counterexample, Left-to-Right Principle.

   A free variable is a variable that has not been bound, drawn from some domain. A predicate is a collection of propositions, indexed by one or more free variables, each drawn from its domain. A counterexample is a specific domain element chosen to make a predicate false. The Left-to-Right Principle states that variables are bound from left to right.

2. Carefully define the following terms: Uniqueness Proof Theorem, Proof by Contradiction Theorem, Proof by Induction Theorem, well-ordered set.

   The Uniqueness Proof theorem states: there is at most one domain element satisfying predicate P if $\forall x, y \in D, P(x) \wedge P(y) \to x = y$. The Proof by Contradiction theorem states: For propositions p, q, if $(p \wedge \neg q) \equiv F$, then $p \to q$ is true. The Proof by Induction theorem states: To prove $\forall x \in \mathbb{N}, P(x)$, we prove (a) $P(1)$ is true; and (b) $\forall x \in \mathbb{N}, P(x) \to P(x+1)$.

3. Simplify $\neg(\exists x, \forall y, \forall z, (x < y) \to (x < z))$ as much as possible (i.e. where nothing is negated). Do not prove or disprove this statement.

   $\forall x, \exists y, \exists z, (x < y) \wedge (x \geq z)$, or (nicer) $\forall x, \exists y, \exists z, z \leq x < y$.

4. Recall that $\mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers. Let $a \in \mathbb{R} \setminus \mathbb{Q}$, $b \in \mathbb{Q}$. Use proof by contradiction to prove that $a + b \in \mathbb{R} \setminus \mathbb{Q}$.

   Because $b \in \mathbb{Q}$, there are integers m, n with $n \neq 0$ and $b = \frac{m}{n}$. Now, assume by way of contradiction that $a+b \in \mathbb{Q}$. Then there are integers s, t with $t \neq 0$ and $a+b = \frac{s}{t}$. We calculate $a = (a+b) - b = \frac{s}{t} - \frac{m}{n} = \frac{sn - mt}{nt}$. Now, $sn - mt, nt$ are integers, and $nt \neq 0$, so $a \in \mathbb{Q}$. This is a contradiction.

5. Prove or disprove: $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (x < y) \to \lceil x \rceil \leq \lfloor y \rfloor$.

   The statement is false. To prove this, we need to prove $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, (x < y) \wedge (\lceil x \rceil > \lfloor y \rfloor)$. Take $x = 0.3, y = 0.4$. We have $x < y$ but $\lceil x \rceil = 1 > 0 = \lfloor y \rfloor$. Many other x, y are possible.

6. Let $n \in \mathbb{Z}$. Use the Division Algorithm to prove that $\frac{(n-1)(n+2)}{2} \in \mathbb{Z}$.

   Apply DA to get integers q, r with $n = 2q + r$, and $0 \leq r < 2$. We now have two cases. If $r = 0$, then $\frac{(n-1)(n+2)}{2} = \frac{(n-1)(2q+0+2)}{2} = (n-1)q \in \mathbb{Z}$. If instead $r = 1$, then $\frac{(n-1)(n+2)}{2} = \frac{(2q+1-1)(n+2)}{2} = q(n+2) \in \mathbb{Z}$.

7. Recall the Fibonacci numbers given by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that for all $n \in \mathbb{N}_0$, $F_{n+2} = 1 + \sum_{i=0}^{n} F_i$.

   Base case $n = 0$: $F_2 = 1 = 1 + F_0 + F_1$. Inductive case: Let $n \in \mathbb{N}_0$ and assume that $F_{n+2} = 1 + \sum_{i=0}^{n} F_i$. Add F_{n+1} to both sides: $F_{n+3} = F_{n+1} + F_{n+2} = F_{n+1} + 1 + \sum_{i=0}^{n} F_i = 1 + \sum_{i=0}^{n+1} F_i$. Hence $F_{n+3} = 1 + \sum_{i=0}^{n+1} F_i$.

8. Let $x \in \mathbb{R}$. Prove that $\lfloor x \rfloor$ exists. That is, prove $\exists n \in \mathbb{Z}$, $n \leq x < n + 1$.

   Let S be the set of all integers less than or equal to x. This is a nonempty set, with an upper bound (x), so by the Maximum Element Induction theorem, there is some maximum element $n \in S$. Since $n \in S$, $n \geq x$. We now prove $x < n + 1$. Assume, by way of contradiction, that $x \geq n + 1$. But then $n + 1 \in S$, and $n + 1 > n$, a contradiction since n was a maximum. Hence $n \leq x < n + 1$.

9. Use induction to prove $\forall n \in \mathbb{N}$, $\frac{(2n)!}{n!n!} \geq 2^n$.

   Base case $n = 1$: $\frac{2!}{1!1!} = 2 \geq 2^1$. Inductive case: Let $n \in \mathbb{N}$, and assume that $\frac{(2n)!}{n!n!} \geq 2^n$. Multiply both sides by $\frac{(2n+1)(2n+2)}{(n+1)(n+1)}$ to get $\frac{(2(n+1))!}{(n+1)!(n+1)!} = \frac{(2n)!}{n!n!} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} \geq 2^n \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = 2^n \frac{(2n+1)2(n+1)}{(n+1)(n+1)} = 2^{n+1} \frac{2n+1}{n+1} \geq 2^{n+1}$. Hence $\frac{(2(n+1))!}{(n+1)!(n+1)!} \geq 2^{n+1}$

10. Let $\mathbb{R}^+$ denote the positive real numbers. Prove that $\forall a \in \mathbb{R}^+$, $\exists b \in \mathbb{R}^+$, $\forall x \in \mathbb{R}$, $|x - 2| < b \to |3x - 6| < a$.

    Let $a \in \mathbb{R}^+$ be arbitrary. Choose $b = \frac{a}{3}$. Now, let $x \in \mathbb{R}$ with $|x - 2| < b$. We have $|x - 2| < b = \frac{a}{3}$. Multiplying both sides by 3, we get $|3x - 6| = 3|x - 2| < a$. Hence $|3x - 6| < a$. This proves that $\lim_{x \to 2} 3x = 6$; to learn much more like this, take Math 534A.